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# BLOCKWISE INTERPOLATION ON SEQUENCES OF POINTS IN THE DISK

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Abstract. This note deals with the classical topic of interpolating bounded sequences of complex numbers by bounded analytic functions on sequences in the unit disk. We introduce linear, polynomial, and multiplicative interpolation over fixed-length consecutive blocks of points taken from these sequences, and examine the corresponding interpolating sequences.

## 1. Introduction and statement of results

Let $\ell^\infty$ be the space of all sequences $\lambda = (\lambda_n)_n$ of complex numbers such that $\|\lambda\|_\infty = \sup_n |\lambda_n| < \infty$. Let $\mathcal{Z} = (z_n)_n$ denote any sequence in the open unit disk $\mathbb{D}$. Let $H^\infty$ be the space of all analytic functions $f$ on $\mathbb{D}$ such that $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty$, and we denote by $\mathcal{B}$ the unit ball of $H^\infty$, i.e., $\mathcal{B} = \{f \in H^\infty : \|f\|_\infty \le 1\}$. The pseudohyperbolic distance between $z$ and $w$ in $\mathbb{D}$ is defined by $\rho(z,w) = \frac{|w-z|}{|1-\overline{w}z|}$. We recall that a sequence $\mathcal{Z}$ is called *uniformly separated* ($US$) if there exists a constant $c > 0$ such that for every $m \in \mathbb{N}$,

$$\prod_{\substack{n\in\mathbb{N}\\ n\neq m}} \rho(z_n, z_m) \ge c\,,$$

and *separated* ($S$) if

$$\inf_{m\neq n} \rho(z_n, z_m) \ge \delta > 0.$$

Clearly, $US$-sequences are separated. However, if $\mathcal{Z}$ lies in a Stolz angle, that is, if there exist $\tau \in \partial\mathbb{D}$ and $\eta \in [1, +\infty)$ such that

$$|z_n - \tau| \le \eta\,(1 - |z_n|) \quad \forall n \in \mathbb{N}\,,$$

then the converse is true (see [2]).

**Definition 1.1.** A sequence $\mathcal{Z}$ is called interpolating if, given any $\lambda \in \ell^\infty$, there exists $f \in H^\infty$ such that $f(z_n) = \lambda_n$ for all $n \in \mathbb{N}$.

Carleson characterized them in 1958 as follows (see [1]).

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**Theorem 1.1.** *$\mathcal{Z}$ is interpolating if and only it is a $US$-sequence.*

Our aim is to add an algebraic component to the topic of analytic interpolation. In this context, we are interested in extending interpolating sequences to evaluation at multiple points of $\mathcal{Z}$ linearly, polynomially and through products In [3], a linear interpolation at two points is considered. Specifically, if $(a_n)_n \in \ell^\infty$, the interpolation is defined by

$$\begin{cases} f(z_1) = \lambda_1 \,, \\ f(z_{n+1}) + a_n f(z_n) = \lambda_{n+1} \quad \forall n \in \mathbb{N} \,. \end{cases}$$

In [4], another linear interpolation at two points is also analysed. Explicitly, let $\Psi : \mathbb{N} \to \mathbb{N}$ be an increasing function, different from the identity. Let $L = \mathbb{N} \setminus \mathrm{Im}\Psi$ and $\mathcal{Z}' = (z_n)_{n \in L}$. Let $\mathcal{R}_{\mathcal{Z}'}$ denote the restriction operator on $\mathcal{Z}'$. Two cases arise: (i) If $L = \{1, \dots, k\}$, then given any $\lambda$ and any set $\delta = \{\delta_1, \dots, \delta_k\}$ of complex numbers, not all equal to zero, there exists $f \in H^\infty$ such that $f(z_{\Psi(n)}) - f(z_n) = \lambda_n$ $\forall n \in \mathbb{N}$, and $\mathcal{R}_{\mathcal{Z}'}(f) = \delta$. (ii) If $L$ is an infinite subsequence, then given any $\lambda$ and any $\mu \in \ell^\infty$, with $\mu \neq (0)_n$, there exists $f \in H^\infty$ such that $f(z_{\Psi(n)}) - f(z_n) = \lambda_n$ $\forall n \in \mathbb{N}$, and $\mathcal{R}_{\mathcal{Z}'}(f) = \mu$.

For our purposes, we introduce the following definitions.

**Definition 1.2.** Let $k \in \mathbb{N}$. We say that a sequence $\mathcal{Z}$ is *$k$-separated* if there exists $\delta > 0$ such that for all $n \in \mathbb{N}$,

$$\rho(z_n, z_{n+k}) \geq \delta \,.$$

**Definition 1.3.** Let $k \in \mathbb{N}$. We say that a sequence $\mathcal{Z}$ is *$k$-spread* if for some $\delta > 0$ and every $n \in \mathbb{N}$, there exists $i \in \{1, \dots, k\}$ with

$$\rho(z_{n+i-1}, z_{n+i}) \geq \delta \,.$$

These notions fit into the following chain of implications:

$$\text{separated} \implies k\text{-separated} \implies k\text{-spread} \implies \text{non-Cauchy.}$$

The first implication is immediate from the definitions. The second follows from the triangle inequality: if $\rho(z_n, z_{n+k}) \geq \delta$ for all $n$, then

$$\delta \leq \rho(z_n, z_{n+k}) \leq \sum_{i=1}^{k} \rho(z_{n+i-1}, z_{n+i}) \,,$$

so at least one term in the sum satisfies $\rho(z_{n+i-1}, z_{n+i}) \geq \delta/k$. For the third, note that a $k$-spread sequence has $\rho(z_m, z_{m+1}) \geq \delta$ for infinitely many $m$, so it cannot be Cauchy. None of the reverse implications hold, as the following examples show.

**Example 1.1** ($k$-separated but not separated)**.** Define $\mathcal{Z}$ by

$$z_n = \frac{1}{2} \cdot (-1)^{\lfloor (n-1)/k \rfloor} \,.$$

This sequence takes the value $\frac{1}{2}$ for $n = 1, \dots, k$, then $-\frac{1}{2}$ for $n = k+1, \dots, 2k$, and continues alternating in blocks of length $k$. Since $z_n$ and $z_{n+k}$ always belong to adjacent blocks, they have opposite signs, and thus $\rho(z_n, z_{n+k}) = \rho\big(\frac{1}{2}, -\frac{1}{2}\big) = \frac{4}{5}$

for all $n$. Hence $\mathcal{Z}$ is $k$-separated. On the other hand, consecutive terms within the same block are equal, so $\rho(z_n, z_{n+1}) = 0$ whenever $n \not\equiv 0 \pmod{k}$. Therefore $\mathcal{Z}$ is not separated.

**Example 1.2** ($k$-spread but not $k$-separated)**.** Define $\mathcal{Z}$ by $z_n = 0$ if $n$ is odd, and $z_n = \frac{1}{2}$ if $n$ is even. Then $\rho(z_n, z_{n+1}) = \rho(0, \frac{1}{2}) = \frac{1}{2}$ for all $n$, so $\mathcal{Z}$ is 1-spread, and hence $k$-spread for every $k \geq 1$. However, $\rho(z_n, z_{n+2}) = 0$ for all $n$, so $\mathcal{Z}$ is not $k$-separated for any even $k$.

**Example 1.3** (Non-Cauchy but not $k$-spread)**.** Let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, and define $\mathcal{Z}$ by $z_n = \left(1 - \frac{1}{n}\right)\zeta$. A direct computation shows that $\rho(z_n, z_{2n}) = \frac{n}{3n-1} \to \frac{1}{3}$ as $n \to \infty$, so $\mathcal{Z}$ is not a Cauchy sequence. However, $\rho(z_n, z_{n+1}) = \frac{1}{2n} \to 0$ as $n \to \infty$, which implies that $\mathcal{Z}$ is not $k$-spread for any $k \in \mathbb{N}$.

Throughout, $k$ denotes a positive integer. Our results on linear and polynomial interpolation are as follows.

**Proposition 1.1.** *Let $a_1, \ldots, a_k \in \mathbb{C} \setminus \{0\}$. If, given any $\lambda \in \ell^\infty$ with $\|\lambda\|_\infty \leq 1$, there exists $f \in \mathcal{B}$ such that*

$$\sum_{i=1}^{k} a_i f(z_{n+i-1}) = \lambda_n \quad \forall n \in \mathbb{N}$$

*or*

$$\sum_{i=1}^{k} a_i f(z_{n+i-1})^i = \lambda_n \quad \forall n \in \mathbb{N}\,,$$

*then $\mathcal{Z}$ is $k$-spread.*

**Proposition 1.2.** *Let $a_1, \ldots, a_k \in \mathbb{C} \setminus \{0\}$ such that $\sum_{i=1}^{k} |a_i| < 1$. If $\mathcal{Z}$ is a US-sequence, then for any $\lambda \in \ell^\infty$, there exists $f \in H^\infty$ such that*

$$f(z_n) + \sum_{i=1}^{k} a_i f(z_{n+i-1}) = \lambda_n \quad \forall n \in \mathbb{N}.$$

**Proposition 1.3.** *Let $a_1, \ldots, a_k \in \mathbb{C} \setminus \{0\}$ such that $\sum_{i=1}^{k} |a_i| i R^{i-1} < 1$ for some $R > 0$. If $\mathcal{Z}$ is a US-sequence, then for any $\lambda \in \ell^\infty$ with*

$$\|\lambda\|_\infty + \sum_{i=1}^{k} |a_i| R^i \leq R\,,$$

*there exists $f \in H^\infty$ such that*

$$f(z_n) + \sum_{i=1}^{k} a_i f(z_{n+i-1})^i = \lambda_n \quad \forall n \in \mathbb{N}.$$

**Remark 1.1.** In Propositions 1.1, 1.2 and 1.3, the finite sums may be replaced by infinite series with coefficients $a_i \in \mathbb{C} \setminus \{0\}$ for all $i \in \mathbb{N}$, satisfying, respectively,

$$\sum_{i=1}^{\infty} |a_i| < \infty, \quad \sum_{i=1}^{\infty} i\,|a_i| < \infty, \quad \text{and} \quad \sum_{i=1}^{\infty} |a_i|\, i R^{i-1} < 1\,.$$

The proofs follow by suitably adapting those of the corresponding propositions.

Regarding the interpolation of products of function values, we state the following results.

**Proposition 1.4.** *If for every $\lambda \in \ell^\infty$ with $\|\lambda\|_\infty \le 1$, there exists $f \in \mathcal{B}$ such that*

$$\prod_{i=1}^{k} f(z_{n+i-1}) = \lambda_n \quad \forall n \in \mathbb{N}, \tag{1}$$

*then $\mathcal{Z}$ is $k$-separated.*

**Proposition 1.5.** *Let $\mathcal{Z}$ be a US-sequence. For any $\lambda \in \ell^\infty$ such that $\lambda_n \neq 0$ for all $n \in \mathbb{N}$, and*

$$\sup_n \frac{|\lambda_{n+1}|}{|\lambda_n|} \le 1, \tag{2}$$

*there exists $f \in H^\infty$ satisfying* (1).

Note that case $k = 1$ in Proposition 1.5, with no further conditions on $\lambda$, corresponds to classical Carleson interpolation. We also consider a mixed interpolation—linear and multiplicative—in the form:

$$\prod_{i=1}^{k} f(z_{n+i-1}) \cdot \sum_{j=1}^{l} a_j f(z_{n+k+j-1}) = \lambda_n \quad \forall n \in \mathbb{N}, \tag{3}$$

where $a_j \in \mathbb{C} \setminus \{0\}$ for all $j = 1, \dots, l$. For the sake of clarity, we restrict ourselves to the simplest case of (3), that is, $k = 1$ and $l = 2$. More precisely, we formulate the interpolation:

$$f(z_n)\big[af(z_{n+1}) + bf(z_{n+2})\big] = \lambda_n \quad \forall n \in \mathbb{N}, \tag{4}$$

where $a, b \in \mathbb{C} \setminus \{0\}$. Note that if $f \in \mathcal{B}$ satisfies (4), then it is necessary that $\|\lambda\|_\infty \le |a| + |b|$. For this mixed interpolation, we have the following results.

**Proposition 1.6.** *Suppose $a, b \in \mathbb{C} \setminus \{0\}$ and*

$$|a| + |b| \le 1 \tag{5}$$

$$(1-\varepsilon)|a| + |b| \ge \frac{|a^2 - b^2|}{|a|} \tag{6}$$

*for some $0 < \varepsilon < 1$. If, given any $\lambda \in \ell^\infty$ with $\|\lambda\|_\infty \le 1$, there exists $f \in \mathcal{B}$, not vanishing on $\mathcal{Z}$, such that* (4) *holds, then $\mathcal{Z}$ is 2-separated.*

For example, the values $a = 0.6$, $b = 0.3$ and $\varepsilon = 0.5$ satisfy conditions (5) and (6). In relation to a criterion ensuring the existence of a function $f \in H^\infty$ satisfying the recursion in (4), we focus on the corresponding numerical equation, namely,

$$x_n\,(a\,x_{n+1} + b\,x_{n+2}) = \lambda_n \quad \forall n \in \mathbb{N}. \tag{7}$$

If $\lambda = (0)_n$, and $x_1$, $x_2 \neq 0$, the solution of (7) is given by $x_{n+2} = -\frac{a}{b}x_{n+1}$, implying that no bounded solution $(x_n)_n$ exists when $a > b$. On the other hand, we have:

**Proposition 1.7.** *If $\lambda \in \ell^\infty$ is such that $|\lambda_n| \geq \delta > 0$ for all $n \in \mathbb{N}$, then* (7) *has no bounded solutions.*

## 2. Proof of results

**Proof of Proposition 1.1.** For the linear interpolation, let

$$f[n] := \sum_{i=1}^{k} a_i f(z_{n+i-1}).$$

By the Schwarz lemma (see [2]), if $f \in \mathcal{B}$, then

$$|f(z_n) - f(z_m)| \leq 2\,\rho(z_n, z_m) \quad \forall n,\, m \in \mathbb{N}\,. \tag{8}$$

Thus,

$$\begin{aligned} |f[n] - f[n+1]| &\leq \sum_{i=1}^{k} |a_i|\, |f(z_{n+i-1}) - f(z_{n+i})| \\ &\leq 2 \sum_{i=1}^{k} |a_i|\, \rho(z_{n+i-1}, z_{n+i})\,. \end{aligned} \tag{9}$$

Assume, for contradiction, that $\mathcal{Z}$ is not $k$-spread. Let $M \geq 4$ be such that

$$M \sum_{i=1}^{k} |a_i| \geq 4\,.$$

For $\varepsilon = \frac{1}{M\sum_{i=1}^{k}|a_i|}$, there exists $n_0 \in \mathbb{N}$ such that $\rho(z_{n_0+i-1}, z_{n_0+i}) < \varepsilon$ for all $i \in \{1, \ldots, k\}$. Define the sequence $\lambda$ by $\lambda_{n_0} = 1$ and $\lambda_n = 0$ for all $n \neq n_0$. If $f \in \mathcal{B}$ is such that $f[n] = \lambda_n$ for all $n \in \mathbb{N}$, then taking $n = n_0$ in (9), we obtain the contradiction $1 < \frac{1}{2}$. Hence, $\mathcal{Z}$ must be $k$-spread.

For the polynomial interpolation, let

$$Pf[n] := \sum_{i=1}^{k} a_i f(z_{n+i-1})^i\,.$$

By (8), we have

$$\begin{aligned} |Pf[n] - Pf[n+1]| &\leq \sum_{i=1}^{k} |a_i|\, |f(z_{n+i-1})^i - f(z_{n+i})^i| \\ &\leq 2 \sum_{i=1}^{k} i\, |a_i|\, \rho(z_{n+i-1}, z_{n+i})\,, \end{aligned}$$

and the proof proceeds analogously. $\square$

**Lemma 2.1.** *Let $a_1, \dots, a_k \in \mathbb{C} \setminus \{0\}$. If $\sum_{i=1}^k |a_i| < 1$, then for every $\lambda \in \ell^\infty$, there exists a unique sequence $(x_n)_n \in \ell^\infty$ such that*

$$x_n + \sum_{i=1}^k a_i x_{n+i-1} = \lambda_n \quad \forall n \in \mathbb{N}\,.$$

**Proof.** Let $x = (x_n)_n \in \ell^\infty$. Define the bounded linear operator $T\colon \ell^\infty \to \ell^\infty$ by

$$(Tx)_n := -\sum_{i=1}^k a_i x_{n+i-1}\,.$$

Then,

$$\|Tx\|_\infty = \sup_n |(Tx)_n| \leq \sum_{i=1}^k |a_i|\,\|x\|_\infty\,,$$

so $\|T\| \leq \sum_{i=1}^k |a_i| < 1$. Thus, the operator $I - T$ is invertible on $\ell^\infty$, and hence surjective. □

**Proof of Proposition 1.2.** By Theorem 1.1, $\mathcal{Z}$ is interpolating. Thus, given any $\lambda \in \ell^\infty$, there exists $f \in H^\infty$ such that $f(z_n) = x_n$ for all $n \in \mathbb{N}$, and we apply Lemma 2.1. □

**Lemma 2.2.** *Let $a_1, \dots, a_k \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \ell^\infty$. If there exists $R > 0$ such that*

$$\sum_{i=1}^k |a_i|\, iR^{i-1} < 1 \quad \text{and} \quad \|\lambda\|_\infty + \sum_{i=1}^k |a_i| R^i \leq R\,,$$

*then there exists a unique sequence $x = (x_n)_n \in \ell^\infty$, with $\|x\|_\infty \leq R$, such that*

$$x_n + \sum_{i=1}^k a_i (x_{n+i-1})^i = \lambda_n \quad \forall n \in \mathbb{N}\,. \tag{10}$$

**Proof.** For all $x \in \ell^\infty$, define the operator $V : \ell^\infty \to \ell^\infty$ by

$$(Vx)_n := -\sum_{i=1}^k a_i\,(x_{n+i-1})^i\,,$$

and let $Tx := \lambda + Vx$. Let $B_R := \{x \in \ell^\infty : \|x\|_\infty \leq R\}$. For any $x, y \in B_R$, we have

$$|(Vx)_n - (Vy)_n| \leq \sum_{i=1}^k |a_i|\, iR^{i-1} |x_{n+i-1} - y_{n+i-1}|\,,$$

so

$$\|Vx - Vy\|_\infty \leq \left(\sum_{i=1}^k |a_i|\, iR^{i-1}\right) \|x - y\|_\infty < \|x - y\|_\infty\,.$$

Thus, $T$ is a contraction on $B_R$. On the other hand, $T(B_R) \subseteq B_R$, since for $x \in B_R$,

$$\|Tx\|_\infty \leq \|\lambda\|_\infty + \|Vx\|_\infty \leq \|\lambda\|_\infty + \sum_{i=1}^k |a_i| R^i \leq R\,.$$

Therefore, $T$ maps $B_R$ into itself. Since $B_R$ is a complete metric space, the Banach fixed point theorem guarantees the existence of a unique fixed point $x \in B_R$ satisfying (10). □

**Proof of Proposition 1.3.** It is the same as in Proposition 1.2 by applying Lemma 2.2. □

**Proof of Proposition 1.4.** Assume that $\mathcal{Z}$ is not $k$-separated. Then, for $\delta = \frac{1}{16}$, there exists $n_0 \in \mathbb{N}$ such that $\rho(z_{n_0}, z_{n_0+k}) < \frac{1}{16}$. Define, for example, the sequence $\lambda \in \ell^\infty$ by $\lambda_{n_0} = \frac{1}{2}$ and $\lambda_n = \frac{1}{4}$ for $n \neq n_0$. Suppose $f \in \mathcal{B}$ satisfies (1). Then, using (8),

$$\begin{aligned}\frac{1}{4} &= |\lambda_{n_0+1} - \lambda_{n_0}| = |f(z_{n_0+k}) - f(z_{n_0})| \prod_{i=1}^{k-1} |f(z_{n_0+i})| \\ &\leq 2\,\rho(z_{n_0}, z_{n_0+k}) < \frac{1}{8}\,,\end{aligned}$$

which is a contradiction. Therefore, $\mathcal{Z}$ must be $k$-separated. □

**Proof of Proposition 1.5.** We seek to construct a sequence $w = (w_n)_n \in \ell^\infty$ such that, if $f \in H^\infty$ satisfies $f(z_n) = w_n$ for all $n \in \mathbb{N}$, then (1) holds. To this end, set

$$w_1 = w_2 = \cdots = w_{k-1} := 1\,, \quad w_k := \lambda_1\,,$$

for $j \geq 2$,

$$w_{kj} := \frac{\lambda_{k(j-1)+1}\,\lambda_{k(j-2)+1}\cdots\lambda_{k+1}}{\lambda_{k(j-1)}\,\lambda_{k(j-2)}\cdots\lambda_k}\,\lambda_1\,,$$

and for $j \geq 1$,

$$w_{kj+i} := \frac{\lambda_{k(j-1)+i+1}\,\lambda_{k(j-2)+i+1}\cdots\lambda_{i+1}}{\lambda_{k(j-1)+i}\,\lambda_{k(j-2)+i}\cdots\lambda_i} \qquad i = 1, \ldots, k-1\,.$$

Let $M := \sup_n \frac{|\lambda_{n+1}|}{|\lambda_n|}$. We have $|w_{kj}| \leq M^{j-1}|\lambda_1|$, and $|w_{kj+i}| \leq M^j$. From (2), it follows that $\|w\|_\infty \leq \max(|\lambda_1|, 1)$. Hence, Theorem 1.1 provides the desired function $f$. □

**Proof of Proposition 1.6.** Assume that $\mathcal{Z}$ is not 2-separated. Then, for the $\varepsilon$ given by (6), there exists $n_0 \in \mathbb{N}$ such that $\rho(z_{n_0+1}, z_{n_0+3}) < \frac{\varepsilon}{2}$. Define the sequence $\lambda \in \ell^\infty$ by $\lambda_{n_0} = |a| + |b|$ and $\lambda_n = 0$ for $n \neq n_0$. Suppose $f \in \mathcal{B}$ satisfies (4). Then, using (8), it follows that

$$\begin{aligned}|a| + |b| &= |\lambda_{n_0+1} - \lambda_{n_0}| = |f(z_{n_0})| \left| \frac{a^2 f(z_{n_0+1}) - b^2 f(z_{n_0+3})}{a} \right| \\ &\leq |a|\,|f(z_{n_0+1}) - f(z_{n_0+3})| + \frac{|a^2 - b^2|}{|a|}\,|f(z_{n_0+3})| < \varepsilon|a| + \frac{|a^2 - b^2|}{|a|}\,.\end{aligned}$$

This contradicts (6), and therefore $\mathcal{Z}$ must be 2-separated. □

**Proof of Proposition 1.7.** If $(x_n)_n$ is a solution of (7) such that $|x_n| \leq C$ for all $n \in \mathbb{N}$, we have

$$|x_n| = \frac{|\lambda_n|}{|ax_{n+1} + bx_{n+2}|} \geq \frac{\delta}{(|a| + |b|)\,C}\,.$$

Then,

$$|x_{n+2}| \leq \frac{|\lambda_n| + |a|\,|x_n|\,|x_{n+1}|}{|b|\,|x_n|} \leq \frac{(\|\lambda\|_\infty + |a|C^2)\,(|a| + |b|)\,C}{|b|\,\delta} := K\,,$$

but it is immediate to verify that there are no parameter values for which $K \leq C$. □

**Remark 2.1.** Despite the results provided herein, the characterization of sequences for which these blockwise algebraic interpolations are possible remains open.

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